Homework 8 (Due Thursday, February 23rd)

1. The canonical partition function for a d-dimensional phonon gas is

$$\ln Q = -\mathcal{C} \int_0^{\omega_m} \ln(1 - e^{-\beta\hbar\omega})\, \omega^{d-1}\, d\omega$$

where $\mathcal{C}$ and ω_m are constants. Show that in the low temperature limit the heat capacity C_v goes as T^d while in the high temperature limit it goes to a constant independent of temperature. Note that while most solids vibrate in three dimensions, the lattices in selenium and graphite roughly behave as one-dimensional and two-dimensional phonon gases, respectively.

2. (a) The canonical partition function for a photon gas is

$$Q = \exp\left[\frac{4\sigma}{3ck} V T^3\right]$$

where σ is Stefan's constant. Find the internal energy $U(T, V)$ and entropy $S(T, V)$.

(b) The energy density of a photon gas is

$$u(\omega) = \frac{a\omega^3}{\exp(\beta\hbar\omega) - 1}$$

where a is a constant. Accurately plot this function to show that it has a peak at $\hbar\omega \approx 2.82\, kT$.

(c) In 1965, Penzias and Wilson discovered a nearly isotropic cosmic microwave background. The spectrum is approximately that of a blackbody at about 2.8 Kelvins with a maximum energy density at a frequency of about $\nu_{\text{max}} = 160$ GHz. In a simple view, this radiation comes from the adiabatic expansion of a much hotter photon cloud that was produced in the Big Bang, about 13 billion years ago. What was the peak frequency, ν_{max}, at the time that life was forming on Earth, about 4 billion years ago? Assume that the radial expansion rate of the universe is constant; note that $\nu = \omega/(2\pi)$.