Homework 6 (Due Thursday, February 16th)

1. Consider two systems, A and B, with states $n_A = 1, \ldots, \Gamma_A$ and $n_B = 1, \ldots, \Gamma_B$.

(a) We define the entropy of system A as

$$S_A = -k \sum_{n_A} \mathcal{P}_A(n_A) \ln \mathcal{P}_A(n_A)$$

and similarly for system B. Show that if we consider the combined system, A and B together, with the assumption that the interaction between the systems A and B is negligible, that entropy is extensive. Specifically, show that by assuming that $\mathcal{P}_{AB}(n_A, n_B) = \mathcal{P}_A(n_A)\mathcal{P}_B(n_B)$ we find that $S_{AB} = S_A + S_B$.

(b) Suppose that we defined the entropy of system A in a more general form as,

$$S_A = -k \sum_{n_A} \mathcal{P}_A(n_A) f(\mathcal{P}_A(n_A))$$

and similarly for system B. Show that the requirement that entropy is extensive means that $f(x) = \ln x$. Again, assume that the interaction between the systems is negligible.

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2. (a) Use the derivation in Appendix C of Pathria and Beale replacing equation (C.4) by the integral,

$$\int_0^\infty e^{-r} r^2 \, dr = 2$$

and show that,

$$V_{3N}(R) = \int \ldots \int \prod_{i=1}^{N} \left(4\pi r_i^2 dr_i\right) = \frac{(8\pi R^3)^N}{(3N)!}$$

where the integrals are constrained to the region of space such that

$$0 \leq \sum_{i=1}^{N} r_i \leq R$$

(b) Use this result to formulate the entropy, $S(U, V, N)$, for an extremely relativistic ideal gas using the micro-canonical ensemble. Specifically, assume that the gas has N particles and the energy of each particle is $\epsilon = pc$ where c is the speed of light. There is no interaction energy between the particles. You may assume that the number of states with energy $E = U$ is approximately the same as the number with $E \leq U$.

(c) Show that in an adiabatic process for this gas PV^γ is constant with $\gamma = 4/3$.

Solutions

1. (a) From the definition of entropy and the assumption of independent probabilities for states in the two systems (i.e., $\mathcal{P}_{AB}(n_A, n_B) = \mathcal{P}_A(n_A)\mathcal{P}_B(n_B)$),

$$\begin{aligned} S_{AB} &= -k \sum_{n_A} \sum_{n_B} \mathcal{P}_{AB}(n_A, n_B) \ln(\mathcal{P}_{AB}(n_A, n_B)) \\ &= -k \left(\sum_{n_B} \mathcal{P}_B(n_B) \right) \left(\sum_{n_A} \mathcal{P}_A(n_A) \ln(\mathcal{P}_A(n_A)) \right) - k \left(\sum_{n_A} \mathcal{P}_A(n_A) \right) \left(\sum_{n_B} \mathcal{P}_B(n_B) \ln(\mathcal{P}_B(n_B)) \right) \end{aligned}$$

but $\sum \mathcal{P} = 1$ (normalization of probabilities when summed over all states) so,

$$\begin{aligned} S_{AB} &= -k \sum_{n_A} \mathcal{P}_A(n_A) \ln(\mathcal{P}_A(n_A)) - k \sum_{n_B} \mathcal{P}_B(n_B) \ln(\mathcal{P}_B(n_B)) \\ &= S_A + S_B \end{aligned}$$

(b) Following the same steps as in part (a),

$$\begin{aligned} S_{AB} &= -k \sum_{n_A} \sum_{n_B} \mathcal{P}_{AB}(n_A, n_B) f(\mathcal{P}_{AB}(n_A, n_B)) \\ &= -k \sum_{n_A} \sum_{n_B} \mathcal{P}_A(n_A)\mathcal{P}_B(n_B) f(\mathcal{P}_A(n_A)\mathcal{P}_B(n_B)) \end{aligned}$$

Working from the other side,

$$\begin{aligned} S_A + S_B &= -k \sum_{n_A} \mathcal{P}_A(n_A) f(\mathcal{P}_A(n_A)) - k \sum_{n_B} \mathcal{P}_B(n_B) f(\mathcal{P}_B(n_B)) \\ &= -k \sum_{n_A} \sum_{n_B} \mathcal{P}_A(n_A)\mathcal{P}_B(n_B) f(\mathcal{P}_A(n_A)) - k \sum_{n_A} \sum_{n_B} \mathcal{P}_A(n_A)\mathcal{P}_B(n_B) f(\mathcal{P}_B(n_B)) \\ &= -k \sum_{n_A} \sum_{n_B} \mathcal{P}_A(n_A)\mathcal{P}_B(n_B) [f(\mathcal{P}_A(n_A)) + f(\mathcal{P}_B(n_B))] \end{aligned}$$

Thus the condition that $S_{AB} = S_A + S_B$ means that $f(\mathcal{P}_A(n_A)\mathcal{P}_B(n_B)) = f(\mathcal{P}_A(n_A)) + f(\mathcal{P}_B(n_B))$, which is the definition of logarithm.

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2. (a) We write $V_{3N}(R) = AR^{3N}$ and $dV_{3N} = A(3NR^{3N-1})dR$. To make use of the given identity we use the integral,

$$\mathcal{A} = \int_0^\infty \ldots \int_0^\infty \exp\left(- \sum_{i=1}^{N} r_i \right) \prod_{i=1}^{N} r_i^2 dr_i = \prod_{i=1}^{N} \int_0^\infty e^{-r_i} r_i^2 dr_i = 2^N \qquad (*)$$

From the definition of $V_{3N}(R)$ the integral on the left may rewritten as,

$$\begin{aligned} \mathcal{A} &= \int_0^\infty e^{-R} (4\pi)^{-N} dV_{3N} = \int_0^\infty e^{-R} (4\pi)^{-N} A(3NR^{3N-1}) dR \\ &= (4\pi)^{-N} A(3N\Gamma(3N)) = \frac{(3N)!A}{(4\pi)^N} \qquad (**) \end{aligned}$$

Matching the expressions in (*) and (**) above we get $A = (8\pi)^N/(3N)!$, which gives the desired result for V_{3N}.

(b) The volume of phase space that we want is such that $E \leq U$ where

$$E = \sum_{i=1}^{N} c p_i$$

Because the energy does not depend on position the integral over the q's gives V^N, as with the standard ideal gas. The volume integral over momenta, with the constraint, is

$$\int \cdots \int \prod_{i=1}^{N} (4\pi p_i^2 dp_i) = \frac{(8\pi U^3/c^3)^N}{(3N)!} \qquad \left(\text{with } \sum_{i=1}^{N} p_i \leq U/c\right)$$

so the entropy is

$$S(U, V, N) = k \ln \Sigma(U, V, N) = k \ln \left(\frac{1}{h^{3N}} V^N \frac{(8\pi U^3/c^3)^N}{(3N)!} \right) = kN \ln(VU^3) + f(N)$$

where $f(N)$ is only a function of the number of particles.

(c) The equation of state is the same as for the standard ideal gas since,

$$\frac{P}{T} = \left(\frac{\partial S}{\partial V}\right)_{U,N} = \frac{kN}{V}$$

so $PV = NkT$. For the temperature we find,

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{V,N} = \frac{3kN}{U}$$

so $U = 3NkT = 3PV$. From this we have that $VU^3 = 27P^3V^4$; from the expression for $S(U, V, N)$ it's clear that entropy is constant if P^3V^4 is constant. Thus for an adiabatic process PV^γ is constant where $\gamma = 4/3$ for the extremely relativistic gas.