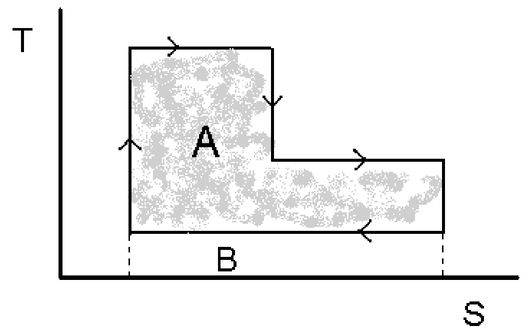
Homework 2 (Due Thursday, February 2nd)

1. An engine is represented by the cyclic transformation shown in the $T - S$ diagram below. The variable A denotes the area of the shaded region and B the area of the region below it. Find the efficiency in terms of the ratio of the areas A and B; show that this engine is not as efficient as a Carnot engine operating between the highest and lowest available temperatures.



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2. Systems $\mathcal{A}$ and $\mathcal{A}'$ are isolated from the rest of the world but are in thermal and mechanical contact with each other (i.e., equal temperature and pressure). The total energy, $U_\Sigma = U + U'$, and total volume $V_\Sigma = V + V'$ are constant. The entropy for system $\mathcal{A}$ is $S(U, V) = k \ln(CU^a V^b)$ Similarly, for system $\mathcal{A}'$, $S'(U', V') = k \ln(C' U'^c V'^d)$ where a, b, c, d, C, and C' are constants. Find U and V, the energy and volume of system $\mathcal{A}$ at thermodynamic equilibrium in terms of U_Σ, V_Σ and the constants a, b, c, d.

Solutions

1. In a cyclic process, the change in the internal energy is zero (i.e., $\Delta U = 0$ for a closed loop). By the first law, this implies that $\Delta Q = \Delta W$. From the definition of entropy,

$$\Delta Q = \oint T\,dS$$

so ΔQ equals the area A.

On the other hand, the heat *absorbed* by the system, Q_+, is the total area below the top segment, thus it equals $A + B$. Note that the heat *removed* from the system, Q_-, equals the area of the lower segment (i.e., the area B).

From the definition of efficiency,

$$\eta = \frac{\Delta W}{Q_+} = \frac{A}{A+B} = \frac{1}{1 + B/A}$$

In the $T - S$ diagram, the Carnot cycle is a rectangle (two isothermal horizontal legs and two adiabatic vertical legs). For the Carnot cycle, the area A is larger so the efficiency η must be larger.

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2. The temperature and pressure for system $\mathcal{A}$ can be obtained from

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_V = \left(\frac{\partial}{\partial U}\right)_V k\ln(CU^aV^b) = \frac{ka}{U}$$

and

$$\frac{P}{T} = \left(\frac{\partial S}{\partial V}\right)_U = \left(\frac{\partial}{\partial V}\right)_U k\ln(CU^aV^b) = \frac{kb}{V}$$

Similarly, for system $\mathcal{A}'$,

$$\frac{1}{T'} = \frac{kc}{U'} = \frac{kc}{U_\Sigma - U}$$

and

$$\frac{P'}{T'} = \frac{kd}{V'} = \frac{kd}{V_\Sigma - V}$$

Since $T = T'$,

$$\frac{ka}{U} = \frac{kc}{U_\Sigma - U}$$

so

$$U = \frac{a}{a+c}U_\Sigma$$

Similarly, since $P/T = P'/T'$,

$$\frac{kb}{V} = \frac{kd}{V_\Sigma - V}$$

so

$$V = \frac{b}{b+d}V_\Sigma$$