Homework 1 (Due Tuesday, January 31st)

Let's get you warmed up with a few exercises on simple gases.

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1. A room (volume $V = 100\text{m}^3$) is initially at a chilly $T_i = 10°$ C. After lighting a fire in the fireplace the temperature in the room rises to $T_f = 25°$ C. The room is not air-tight so the initial and final pressures are one atmosphere. Taking the air to be an ideal gas with a heat capacity per molecule of $c_V = \frac{5}{2}k$, find the initial and final energies, U_i and U_f, for the air in the room.

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2. Consider n moles of an ideal gas undergoing an adiabatic process (i.e., Q constant). Show that PV^γ is constant, where $\gamma = c_P/c_V$ is the ratio of the ratio of the heat capacities per molecules. Note that for an ideal gas $C_P - C_V = Nk$.

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3. The equation of state for a hard sphere gas is

$$P = \frac{nRT}{V - bn}$$

where b is a constant. Note that we recover the ideal gas law when $b = 0$. As with an ideal gas, for the hard sphere gas the internal energy only depends on the temperature so $U = C_V T$ and $C_P - C_V = Nk$. If n moles of a hard sphere gas, initially at pressure P_0 and volume V_0, undergoes an adiabatic process then $P = f(V)$. Obtain an expression for the function $f(V)$.

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4. The Ruchhardt experiment is a classic method to measure $\gamma = c_P/c_V$ for an ideal gas. In the experiment a large bottle of volume V_0, filled with gas at atmospheric pressure P_0, is fitted with a stopper cork with a glass tube. In the tube is a small metal ball (radius r, density ρ) that fits snugly but moves freely (like a piston); at rest it is supported by the pressure in the bottle. When displaced a small distance from rest the ball moves up and down the tube with simple harmonic motion. Assuming that the process is adiabatic use the result obtained in Exercise 2 to find an expression for the oscillation frequency, f, in terms of γ and the other parameters in the experiment. [Hint: You should find that $f \propto \sqrt{r}$.]

Solve these problems yourself or in discussion with your classmates.

DO NOT TRY LOOKING UP THE SOLUTIONS ON THE INTERNET.

Solutions

1. The internal energy of an ideal gas is

$$U = C_V T = c_V NT = \frac{5}{2} kNT = \frac{5}{2} PV$$

using the ideal gas law, $PV = NkT$. The initial and final pressures are $P_i = P_f = 1$ atm $= 1.01 \times 10^5$ Pascals so $U_i = U_f = 2.525 \times 10^7$ Joules. The number of air molecules in the room is $N = PV/kT$ so N decreases as T increases and thus the energy per particle increases. However all the energy added by the fire leaves the room with the particles that leave the room.

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2. There are a variety of ways to solve this problem. We can start with the first law,

$$dQ = dU + PdV$$

In an ideal gas, $U = C_V T$ so $dU = C_V dT$ and thus

$$dQ = C_V dT + PdV \qquad (*)$$

Using the ideal gas law, $PV = NkT$, so for an infinitesimal process we have

$$d(PV) = d(NkT)$$

Since N is fixed and k is a constant,

$$PdV + VdP = Nk(dT)$$

so

$$\begin{aligned} dQ &= C_V dT + (NkdT - VdP) \\ &= (C_V + Nk)dT - VdP = C_P dT - VdP \qquad (**) \end{aligned}$$

For an adiabatic process ($dQ = 0$), we may write $(*)$ and $(**)$ as,

$$PdV = -C_V dT \qquad \text{and} \qquad VdP = C_P dT$$

Dividing the latter equation by the former,

$$\frac{dP}{P} = -\frac{C_P}{C_V}\frac{dV}{V} = -\gamma\frac{dV}{V}$$

Note that $C_P/C_V = c_P/c_V$. Integrate both sides and you get,

$$\ln P = -\gamma \ln V + a$$

where a is the constant of integration. From this we have

$$PV^\gamma = A$$

where $A = e^a$ is a constant.

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3. This problem is very similar to the previous problem. Again, we can start with the first law,

$$dQ = dU + PdV$$

As before, $U = C_V T$ so $dU = C_V dT$ and thus

$$dQ = C_V dT + PdV \qquad\qquad (*)$$

Using the hard sphere gas law, $PV' = NkT$, where $V' = V - bn$; note that $dV' = dV$ since b and n are constant. So for an infinitesimal process we have

$$d(PV') = d(NkT)$$

Since N is fixed and k is a constant,

$$PdV + V'dP = Nk(dT)$$

so

$$\begin{aligned} dQ &= C_V dT + (NkdT - V'dP) \\ &= (C_V + Nk)dT - V'dP = C_P dT - V'dP \qquad\qquad (**) \end{aligned}$$

For an adiabatic process ($dQ = 0$), we may write $(*)$ and $(**)$ as,

$$PdV = -C_V dT \qquad \text{and} \qquad V'dP = C_P dT$$

Dividing the latter equation by the former,

$$\frac{dP}{P} = -\frac{C_P}{C_V}\frac{dV}{V'} = -\gamma \frac{dV'}{V'}$$

Note that $C_P/C_V = c_P/c_V$. Integrate both sides and you get,

$$\ln P = -\gamma \ln V' + a$$

where a is the constant of integration. From this we have

$$P = A(V - bn)^{-\gamma}$$

where $A = e^a$ is a constant. This constant may be fixed by the initial state so

$$P = P_0 \frac{(V_0 - bn)^\gamma}{(V - bn)^\gamma}$$

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4. Using the result from Exercise 2, for an adiabatic process $P = \mathcal{C}V^{-\gamma}$ where $\mathcal{C}$ is a constant. Since the pressure supports the ball we may write the force on the ball as $F = PA = \mathcal{C}AV^{-\gamma}$ where $A = \pi r^2$ is the cross-sectional area of the ball (and of the tube). When the ball's position moves the volume changes so the force changes and we may this as

$$dF = -\gamma \mathcal{C}AV^{-\gamma-1}dV = -\gamma PAV^{-1}dV$$

The change in volume is related to the ball's displacement, dx, from its equilibrium position as $dV = Adx$ so

$$dF = -\gamma PA^2V^{-1}dx$$

Since the oscillations have simple harmonic motion the force obeys Hooke's law so $dF/dx = -k$ so the effective spring constant in the Ruchhardt experiment is $k = \gamma P_0 A^2 V_0^{-1}$. Note that since the oscillations are small we use the equilibrium values for pressure and volume. Finally, from freshman physics we know that the frequency of a spring oscillator is $f = (2\pi)^{-1}\sqrt{k/m}$ so our final result is

$$f = \frac{1}{2\pi}\sqrt{\frac{\gamma P_0 \pi^2 r^4}{mV_0}} = \frac{1}{2\pi}\sqrt{\frac{3\gamma P_0 \pi r}{4\rho V_0}}$$

since $m = \frac{4}{3}\pi r^3 \rho$. Note that a small correction may be made in this expression by replacing P_0 with $P_0 + mg/A$ to account for the fact that the pressure in the bottle at equilibrium is slightly higher than atmospheric due to the weight of the ball.