Homework 13 (Due Tuesday, March 20th)

1. Show that the equation of state for the nondegenerate ($g_d = 1$) ideal Bose gas may be written in the form of a virial expansion,

$$\frac{Pv}{kT} = 1 - \frac{1}{4\sqrt{2}}\left(\frac{\lambda^3}{v}\right) + \left(\frac{1}{8} - \frac{2}{9\sqrt{3}}\right)\left(\frac{\lambda^3}{v}\right)^2 - \dots$$

where $v = V/N$ is the volume per particle; see eqns. (7.1.13,14) in Pathria and Beale. Note that this is a high temperature expansion so you may neglect the low temperature ground state correction (i.e., the term that contributes when $z \approx 1$). Hint: Use the series representations of the Bose integrals $g_{3/2}$ and $g_{5/2}$.

2. Consider a two-dimensional ideal Bose gas (i.e., particles are on a surface).

(a) Show that

$$\ln \mathcal{Q} = \frac{Ag_d}{\lambda^2} g_2(z) - g_d \ln(1-z)$$

where $\mathcal{Q}$ is the grand canonical partition function, A is the area of the container (i.e. two-dimensional "volume"), g_d is the degeneracy factor for the energy levels and g_2 is a Bose integral.

(b) Show that the number of particles in the excited states diverges as $z \to 1$ for $T \neq 0$ (i.e., no Bose-Einstein condensation for a two dimensional boson gas).

3. (a) Show that for a 2-dimensional Fermi gas in an area A,

$$\frac{N}{A} = \frac{g_d}{\lambda^2} f_1(z) = \frac{g_d}{\lambda^2} \ln(1+z)$$

and for a 2-dimensional Bose gas,

$$\frac{N}{A} = \frac{g_d}{\lambda^2} g_1(z) = \frac{g_d}{\lambda^2} \ln(1-z)$$

Note that there is no Bose-Einstein condensation in the 2-D Bose gas.

(b) Show that for a 2-dimensional Fermi gas in an area A,

$$U = \frac{g_d AkT}{\lambda^2} f_2(z)$$

and that the corresponding result for a 2-D Bose gas is,

$$U = \frac{g_d AkT}{\lambda^2} g_2(z)$$

4. (a) Show that the isothermal compressibility, K_T, for a Bose gas above the critical temperature is

$$K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N} = \frac{V}{NkT}\frac{g_{1/2}(z)}{g_{3/2}(z)}$$

Hint: Use the series representation to evaluate derivatives.

(b) The standard deviation in the number of particles in a volume V depends on the compressibility, specifically,

$$\sigma_N = \sqrt{\langle N^2 \rangle - \langle N \rangle^2} = N\sqrt{\frac{kTK_T}{V}}$$

Find σ_N for a Bose gas in the high temperature (i.e., non-quantum) limit.

(c) Obtain an expression for σ_N in the low temperature limit as T approaches the critical temperature (i.e., condensation temperature) from above. Your expression should contain z but not $g_\nu(z)$ (take the appropriate limit). Describe the behavior of σ_N in this limit. Hint:

$$g_\nu(e^{-\alpha}) \approx \frac{\Gamma(1-\nu)}{\alpha^{1-\nu}} \qquad 0 \leq \nu < 1, \quad 0 < \alpha \ll 1$$