Supplementary Notes for *Physics of Timing and Spacing*

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June 22, 2008

The physics in the *Physics of Timing and Spacing* tutorial is linear kinematics, as discussed in an early chapter of any standard introductory physics textbook. These notes summarize the mathematical basis for the results presented in that tutorial.

Besides seconds we'll use a frame (fr) as a unit of time with one frame of film equalling $1/24^{\text{th}}$ of a second. This is the most commonly used frame rate although others are also in use (e.g., 30 fps for video).

In linear kinematics, the distance traveled by an object under constant acceleration, initially at position x_1 with initial velocity v_1, is

$$x(t) = x_1 + v_1 t + \tfrac{1}{2} a t^2 \tag{1}$$

The time will be expressed in terms of frame number, n, with $t = 0$ being frame 1; the distance traveled is

$$x_n = x_1 + v_1(n-1) + \tfrac{1}{2} a (n-1)^2 \tag{2}$$

For the case of the falling ball the acceleration is $a = g$, which we take as $g = \frac{2}{3}$ in/fr^2 (or 32 ft/s^2). Also taking the initial state as a release of the ball from rest, the distance fallen is

$$x_n = \tfrac{1}{2} g (n-1)^2 = (\tfrac{1}{3}\text{ inch})(n-1)^2 \tag{3}$$

The "Distance Fallen" table is easily reproduced using this formula (note that the distance fallen after n' elapsed frames is x_n with $n = n' + 1$).

In animation not every frame has a different drawing, for example when "shooting on twos" each drawing is repeated for two frames. Call S the number of repeated drawings (e.g., shooting on twos is $S = 2$ frames per drawing) then the position of drawing s is

$$x_{(s)} = x_1 + v_1 S (s-1) + \tfrac{1}{2} a S^2 (s-1)^2 \qquad s = 1, 2, \ldots \tag{4}$$

The basis for the "Odd Rule" is the simple observation that the spacing between consecutive drawings is

$$\Delta x_{(s)} = x_{(s+1)} - x_{(s)} = v_1 S + \tfrac{1}{2} a S^2 (s^2 - (s-1)^2) = v_1 S + \tfrac{1}{2} a S^2 (2s - 1) \tag{5}$$

so from the apex (i.e., $v_1 = 0$), the ratios of the spacings between drawings is

$$\frac{\Delta x_{(s+1)}}{\Delta x_{(s)}} = \frac{2s+1}{2s-1} \tag{6}$$

Thus $\Delta x_{(1)} : \Delta x_{(2)} : \Delta x_{(3)} : \ldots$ are in the ratios $1 : 3 : 5 : \ldots$. Notice that the Odd Rule gives the ratios of the spacings from an apex regardless of the value of the (constant) acceleration.

The increment version of the Odd Rule is from,

$$\Delta x_{(s+1)} - \Delta x_{(s)} = \tfrac{1}{2} a S^2((2s+1) - (2s-1)) = aS^2 \tag{7}$$

Note that the increment version Odd Rule applies even when the first drawing is not the apex. If the first drawing is the apex then the first spacing, $\Delta x_{(1)} = \frac{1}{2}aS^2$, equals half the increment between consecutive drawings.

The "Fourth Down at Half Time" rule comes from the observation that from the apex, the distances fallen from that apex at times $t = T/2$ and $t = T$ are

$$x(T/2) - x(0) = \tfrac{1}{2}a(T/2)^2 = \tfrac{1}{4}\left(\tfrac{1}{2}aT^2\right) \tag{8}$$

$$x(T) - x(0) = \tfrac{1}{2}aT^2 \tag{9}$$

Note that this rule applies in general for motion with constant acceleration as long as the distances are measured from a position with initial velocity zero.

To locate the in-betweens, notice that the distance from the apex for $S = 2$ is

$$x_{(s)} = \tfrac{1}{2}a(4)(s-1)^2 = \tfrac{1}{2}a(2s-2)^2 \tag{10}$$

When $S = 1$ the matching drawing to s is $s^* = 2s - 1$ so the drawing in between s and $s+1$ is $s^* = 2s$, which is at a distance

$$x_{(s^*)} = \tfrac{1}{2}a(s^*-1)^2 = \tfrac{1}{2}a((2s-2)+1)^2 = x_{(s)} + \tfrac{1}{2}a(4s-3) = x_{(s)} + \left(\frac{4s-3}{8s-4}\right)\Delta_{(s)} \tag{11}$$

Notice that for $s = 1$, $x_{(s^*=2)} = x_{(s=1)} + \frac{1}{4}\Delta_{(s=1)}$, that is, the first in-between drawing from the apex is a quarter of the distance from the apex. Also notice that for large values of s we have $x_{(s^*)} \approx x_{(s)} + \frac{1}{2}$ so the farther the object is from the apex the closer the in-betweens are to being in the center of the interval.

Finally, the table of reaction times is most easily computed using $t = \sqrt{2gx(t)}$.